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Elastohydrodynamic collisions of solid spheres

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Recent developments in solving the problem of the elastohydrodynamic collision between two solid elastic bodies involved elaborate numerical procedures in order to simultaneously account for the elastic deformation of the solid surfaces and viscous fluid pressure. This paper describes a simple analytical approximation based upon a Hertzian-like profile for the elastic deformation of the two solid elastic spheres. By introducing a scaling coefficient, a closed-form solution has been developed which is capable of predicting the evolution of the relative particle velocity, force and restitution coefficient to an accuracy that is comparable with the exact numerical solutions.

1. Introduction

The viscous force arising from the radial pressure flow of the interstitial fluid between two colliding solid particles is of particular relevance to powder dispersion, mixing, and aggregation. Historically, elastohydrodynamics has been concerned with the case of impacting lubricated bodies and has received considerable attention in tribology (e.g. Briscoe & McClune 1976; Safa & Gohar 1986; Larsson & Lundberg 1994). Recently, researchers in the field of filtration, coagulation and adhesion have also examined the elastohydrodynamic collisions of spherical particles. For the case of a pendular liquid bridge between two rigid spheres, a solution based on the lubrication approximation for the viscous force was derived by Adams & Perchard (1985). Their solution is similar to the asymptotic solution for a rigid sphere in a viscous fluid moving towards a rigid wall (Cox & Brenner 1967). However, for deformable solids, the viscous force is affected by the dynamic deformation of the solids. Therefore, it is necessary to account for the coupling between the equations of solid mechanics and fluid dynamics.

Ennis *et al.* (1990) proposed a simplified model in which the rigid-sphere solution was employed and an arbitrary minimum separation distance was specified. This arbitrary separation distance was assumed to correspond to the surface roughness of granules. More rigorous studies of the elastohydrodynamic collision of two spheres that are surrounded by thin isoviscous liquid layers were made by Davis, Serayssol & Hinch (1986) and others (Serayssol & Davis 1986, Barnocky & Davis 1988*a*). However, they showed that analytical solutions for the dynamic deformation and viscous force are not available. The radial pressure of the interstitial fluid developed between the colliding particles may only be solved numerically. Recently, Wells (1993) proposed a more simplified model where the dynamic deformation of the spheres was assumed to be Hertzian-like. The results showed some agreement with the data of Davis *et al.* (1986), but it was still necessary to obtain the solution numerically. There was also significant divergence from the exact solution due to Davis *et al.* (1986) for large Stokes numbers, as will be discussed later.

The intention of the current paper is to present a simplified analytical model for the isoviscous elastohydrodynamic collision of deformable solid spheres. The trajectories and viscous fluid forces of the deformable particles are predicted and scaled to the accurate numerical results of Davis *et al.* (1986). The approach adopted is analogous to that of Wells (1993) in that a Hertzian-like profile for the elastic deformation of the solid spheres is employed. The development of a closed-form solution is important in the efficient implementation of simulation techniques involving many-body problems. For example, squeeze film interactions have been identified as being dominant in the flow of concentrated suspensions (Frankel & Acrivos 1967). In order to account for such hydrodynamic interactions a number of numerical procedures have been developed including Stokesian dynamics (Bossis & Brady 1989). The current approaches are limited to rigid-body interactions, but there is a growing interest in soft or deformable particles for which the elastohydrodynamic interactions become important. Another field of interest is granulation whereby agglomerates of particles having liquid bridges collide to form larger agglomerates (Ennis, Tardos & Pfeffer 1991). Some progress has been made using the discrete element method to simulate this process (Lian, Thornton & Adams 1993; Thornton, Lian & Adams 1993).

2. Theoretical considerations

When two elastic spheres separated by a thin layer of liquid or a pendular liquid bridge collide with each other along the line-of-centres, a radial pressure is developed within the interstitial fluid. According to elasticity theory, if the radial pressure is denoted by $p(r)$, the elastic deformation of the surfaces of the spheres, $w_i(r)$, is then given by (Davis *et al.* 1986)

$$w_i(r) = \frac{2(1-\nu_i^2)}{E_i}\int_0^\infty p(\xi)\frac{\xi}{\xi+r}K\left[\frac{4r\xi}{(\xi+r)^2}\right]\mathrm{d}\xi, \tag{1}$$

where K is the complete elliptic integral of the first kind, E_i and ν_i are the elastic properties of the spheres ($i = 1$ or 2) and the radial coordinates ξ and r are shown in figure 1. The elastic deformation, $w_i(r)$, is related to the separation distance between the spheres, $h(r)$, by the following expression:

$$h(r) = S + r^2/2R^* + w_r, \tag{2}$$

where $w_r = w_1(r) + w_2(r)$ and the parameter S is the instantaneous separation distance at $r = 0$ between the two spheres in their undeformed state (see figure 1) and R^* is the reduced radius expressed in terms of the radii of the two spheres ($1/R^* = 1/R_1 + 1/R_2$). The instantaneous value of S is described by the kinematic equations of relative motion which may be expressed as

$$\dot{S} = v, \quad \dot{v} = F_v/m^*, \tag{3}$$

where v is the relative velocity of approach, F_v is the viscous force and $m^*(1/m^* = 1/m_1 + 1/m^2)$ is the reduced mass; the dot refers to differentiation with respect to time.

In order to determine the dynamic normal deformation, the hydrodynamic pressure in the fluid needs to be specified. For the relative displacement of two spheres in close contact, according to the classical lubrication theory, the radial pressure distribution of a Newtonian fluid should satisfy the equation

$$rh^3(r)\frac{\partial p(r)}{\partial r} = 12\eta\int_0^r \dot{h}(r)\,r\,\mathrm{d}r, \tag{4}$$

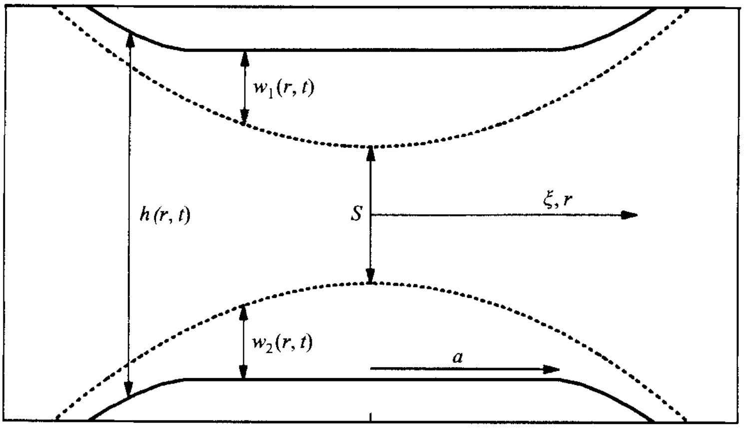

FIGURE 1. Schematic representation of the dynamic deformation of two elastic spheres separated by an interstitial fluid.

where η is the fluid viscosity. From the radial pressure distribution, the viscous force may be derived as

$$F_v = \int_0^\infty 2\pi p(r)\, r\, \mathrm{d}r. \tag{5}$$

It may be seen that the elastic and hydrodynamic equations are fully coupled. For relatively hard spheres or large separation distances, when the elastic deformation is very small, the gap $h(r)$ may be approximated as

$$h(r) = S + r^2/2R^* \tag{6}$$

from which the viscous force may be obtained as

$$F_v = 6\pi\eta R^* v \frac{R}{S}. \tag{7}$$

Equation (7) is the rigid-sphere solution derived by Adams & Perchard (1985), which is the leading-order term that was obtained by Cox & Brenner (1967). However, for relatively soft spheres and small separation distances, the elastic deformation significantly modifies the viscous force as described in the introduction. Davis *et al.* (1986) showed that the coupled equations cannot be solved analytically. In an attempt to reduce the complexity of the numerical procedures required, Wells (1993) assumed a Hertzian-like profile for the elastic deformation of the spheres as defined by the expressions

$$w_{r<\rho_1} = \alpha - \frac{r^2}{2R^*}, \quad w_{r>\rho_1} = \frac{4\alpha}{3\pi} \frac{a}{r-\rho_c}, \tag{8}$$

where $\alpha(= a^2/R^*)$ is the deformation at the centreline and a is the radius of the Hertzian contact area. According to Wells (1993), ρ_c is chosen to be $0.1a$ and $\rho_1 = 1.05343a$. He showed that the above approximation leads to a fourth-order system of ordinary differential equations which, unfortunately, still cannot be solved analytically. He also indicated that the approximation leads to an inconsistency in the pressure distributions for the elastic deformation and viscous fluid equations. However, he was able to demonstrate that the numerical solution was generally in good agreement with that of Davis *et al.* (1986) except for large Stokes numbers as mentioned in the introductory section.

3. Approximate analytical solution

For the elastohydrodynamic collision of two elastic spheres along the line of their centres, Davis *et al.* (1986) have performed an exact numerical analysis. They found that during close approach, the spheres deform such that a central flattened region was developed, which we will approximate to a planar geometry. Subsequently, this will be referred to as the 'inner region' while that at greater values of the radial coordinate will be termed the 'outer region'. Davis *et al.* (1986) found that as the gap between the surfaces decreased, the radius of the inner region increased. This radius reached a maximum value and then decreased as relaxation occurred and the spheres began to rebound. The planar profile developed by the hydrodynamic pressure in the inner region is quite similar in form to the Hertzian deformation for unlubricated solid sphere collisions when the pressure distribution is given by (see Johnson 1985)

$$p(r) = p_0(1-r^2/a^2)^{1/2} \tag{9}$$

and the elastic deformation force between the spheres, F_e, is given by

$$F_e = \frac{4E^*}{3R^*}a^3 = \frac{4}{3}E^*(R^*\alpha^3)^{1/2}, \tag{10}$$

where E^* is related to Young's modulus (E_i), and the Poisson ratio (ν_i) of the two spheres,

$$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1}+\frac{1-\nu_2^2}{E_2}. \tag{11}$$

In accordance with Hertzian theory, the elastic deformation of the inner region ($r < a$) is expressed as

$$w_{r<a} = \frac{a^2}{R^*}-\frac{r^2}{2R^*} \tag{12}$$

and that outside ($r > a$) as

$$w_{r>a} = \frac{1}{\pi R^*}\left[a(r^2-a^2)^{1/2}+(2a^2-r^2)\arcsin\frac{a}{r}\right]. \tag{13}$$

If a Hertzian deformation is assumed, it follows that in the inner region the gap between the two surfaces may be derived by the substitution of equation (12) into (2); thus

$$h_{r<a} = S+a^2/R^* = S+\alpha. \tag{14}$$

The separation distance defined by (14) is constant across the inner region.

Further substitution of (14) into (4) yields the dynamic pressure in the inner region

$$p_{r<a} = C-3\eta(v-\dot{\alpha})\,r^2/h^3, \tag{15}$$

where C is an integration constant. Integration of (15) leads to the following expression for the viscous force:

$$F_{r<a} = \pi R^*\alpha C-\tfrac{3}{2}\pi\eta R^*(v-\dot{\alpha})\frac{R^*\alpha^2}{(S+\alpha)^3}. \tag{16}$$

For the outer region, the elastic deformation given by (13) may be approximated by a truncated Taylor series given as

$$w_{r>a} \approx \frac{4a^3}{3\pi R^* r}. \tag{17}$$

This is quite similar in form to (8), the elastic deformation profile proposed by Wells (1993), if we let $\rho_c = 0$ and $\rho_1 = a$. However, after substituting (17) into (2) and (4), it is not possible to derive an analytical expression for the viscous force in the outer region.

Both the Hertzian pressure distribution (equation (9)) and that computed by Davis *et al.* (1986) are parabolic. However, in the case of the former, the pressure is zero at $r = a$ while, for the latter, the distribution extends to the outer region ($r > a$). Consequently, the Hertzian equation will underestimate the deformation in this region. The effect is relatively modest since the deformation is small compared both to that in the inner region and to the gap between the two surfaces in the outer region where the leading-order term is $r^2/2R^*$. Given that the geometry of the outer region has proved inaccessible by analytical means, the following simplification was introduced:

$$w_{r>a} \approx c_k \alpha = c_k a^2/R^*, \tag{18}$$

where c_k would be equal to 0.5 in order to maintain the continuity of deformation at $r = a$ (see (12) and (18)). However, here we employ c_k $(0 < c_k < 0.5)$ as an adjustable scaling coefficient to be discussed later. In order to maintain the continuity in the geometry of the deformed surfaces, the inner region is extended such that $r \leqslant a_1$ $(a \leqslant a_1)$. Consequently, within the extended inner region

$$w_{r<a_1} = \frac{a^2}{R^*} - \frac{r^2}{2R^*} \tag{19}$$

and outside the extended inner region

$$w_{r>a_1} \approx c_k \alpha = c_k a^2/R^*, \tag{20}$$

where the radius of the extended inner region a_1 is given by the relationship

$$a_1 = a[2(1-c_k)]^{1/2}, \tag{21}$$

such that $a_1 = a$ when $c_k = 0.5$.

Equation (16) now may be modified to account for the change in the radius of the inner region; thus

$$F_{r<a_1} = 2\pi(1-c_k) R^* \alpha C - 6\pi(1-c_k)^2 \eta R^* (v-\dot{\alpha}) \frac{R^* \alpha^2}{(S+\alpha)^3}. \tag{22}$$

In the outer region, the separation distance is now expressed as

$$h_{r>a_1} = S + c_k \alpha + r^2/2R^*, \tag{23}$$

where the elastic deformation for $r > a_1$ is only significant as $r \to a_1$. After substitution of (23) into (4), the viscous pressure in the outer region becomes

$$p_{r>a_1} = 3\eta(v - c_k \dot{\alpha}) R^*/h^2. \tag{24}$$

Comparing (15) and (24) with (9) it is found that the pressure distribution derived for the fluid is not consistent with the Hertzian distribution for the elastic deformation. In fact, by proposing an approximation for the elastic deformation of the spheres, it is impossible to obtain a consistent dynamic pressure distribution which everywhere satisfies both the elasticity and lubrication equations. However, the consistency in the total force as a function of the relative approach and velocity may be satisfied. Integration of (24) leads to

$$F_{r>a_1} = 6\pi\eta R^*(v - c_k\dot{\alpha})\, R^*/(S+\alpha). \tag{25}$$

However, the continuity of the dynamic pressure defined by (15) and (24) requires that at $r = a_1$

$$C = \frac{3\eta R^*(v - c_k\dot{\alpha})}{(S+\alpha)^2} + \frac{6\eta\alpha R^*(v-\dot{\alpha})}{(S+\alpha)^3}(1-c_k). \tag{26}$$

Substitution of (26) into (22) leads to the following expression for the total viscous force:

$$\begin{aligned} F_v &= F_{r<a_1} + F_{r>a_1} \\ &= 6\pi\eta R^*\left[(v - c_k\dot{\alpha})\frac{R^*[S+(2-c_k)\,\alpha]}{(S+\alpha)^2} + (v-\dot{\alpha})\frac{R^*\alpha^2}{(S+\alpha)^3}(1-c_k)^2\right]. \end{aligned} \tag{27}$$

Using the relationship $F_v = F_e$, a much simplified analytical solution for the elastohydrodynamic collision of spherical solids is thus obtained.

4. Results and discussion

Davis *et al.* (1986) presented some numerical results for the elastohydrodynamic collisions of two spherical solids covered by a thin layer of interstitial incompressible Newtonian fluid for which the pressure coefficient of the viscosity was zero. They introduced two dimensionless parameters. The first was the Stokes number which was defined as

$$St = \frac{m^* v_0}{6\pi\eta(R^*)^2}, \tag{28}$$

where v_0 is the initial relative velocity of the two spheres. The second was the elasticity parameter defined as

$$\epsilon = \frac{4\eta v_0 (R^*)^{3/2}}{\pi E^* S_0^{5/2}} \tag{29}$$

where S_0 is the initial separation distance at $r = 0$ between the two colliding spheres in their undeformed state. Similar results have been computed with the present model. Starting from the initial values of a given separation distance and relative velocity, the initial force was obtained using (7). An implicit time-stepping scheme was then applied to compute the instantaneous values of the separation distance, relative velocity, viscous force and elastic deformation using (3), (10) and (27) respectively.

In figure 2, the evolution of the relative particle velocity predicted by the present analytical model is compared with the accurate numerical solution of Davis *et al.* (1986) (referred to below as DSH) for the case of $\epsilon = 0.01$ with $St = 5$. The predictions given by Wells (1992) are also plotted. It can be seen that compared with DSH, when the scaling coefficient c_k is set to 0.5, the present analytical model predicts an under-damped relative velocity, which is in contrast to the over-damped relative velocity predicted by Wells. This may suggest that, by setting $c_k = 0.5$ ($w_{r>a_1} = 0.5\alpha$) for the present model, i.e. assuming that the elastic deformation in the outer region is half that at the centre, the overall elastic deformation is overestimated. By reducing the

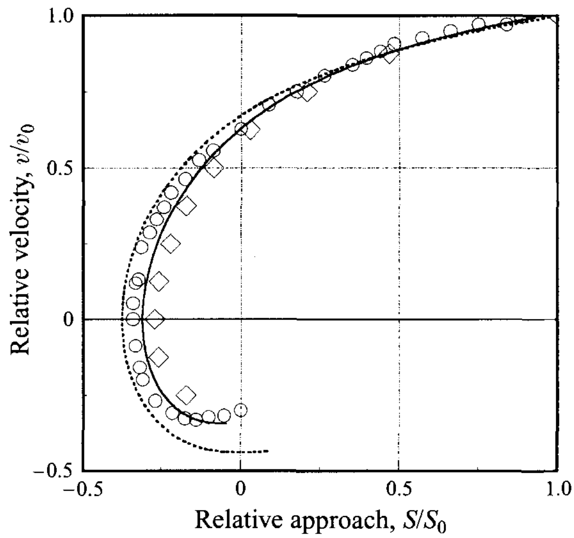

FIGURE 2. Evolution of relative particle velocities predicted by the present model compared with the DSH (○) and Wells (◇) solutions for an elasticity parameter $\epsilon = 0.01$ and Stokes number $St = 5$ and for c_k values of 0.5 (---) and 0.25 (——).

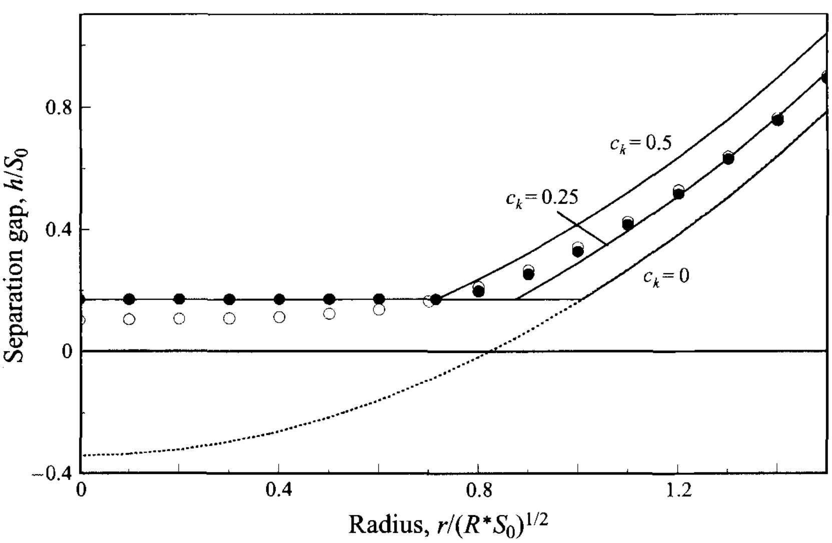

FIGURE 3. Deformed-gap profile of the sphere predicted by the present model (——) compared with the DSH solution (○), the corresponding Hertzian deformation (●) and the undeformed profile (----) for $St = 5$, $\epsilon = 0.01$ and $v_0 t/S_0 = 2$.

scaling coefficient to $c_k = 0.25$ ($w_{r>a_1} = 0.25\alpha$), which is equivalent to reducing the elastic deformation to one quarter of that at the centre, figure 2 indicates that the predicted relative velocity leads to a slightly over-damped solution, but an improved fit to the accurate DSH solution. In fact, $c_k = 0.25$ is the optimum value which minimizes the difference in the maximum rebound velocity between the present model and the DSH solution. it was found that any further reduction in the scaling coefficient results in an over-damped relative velocity.

In figure 3, the deformed profiles for different c_k values are plotted and compared

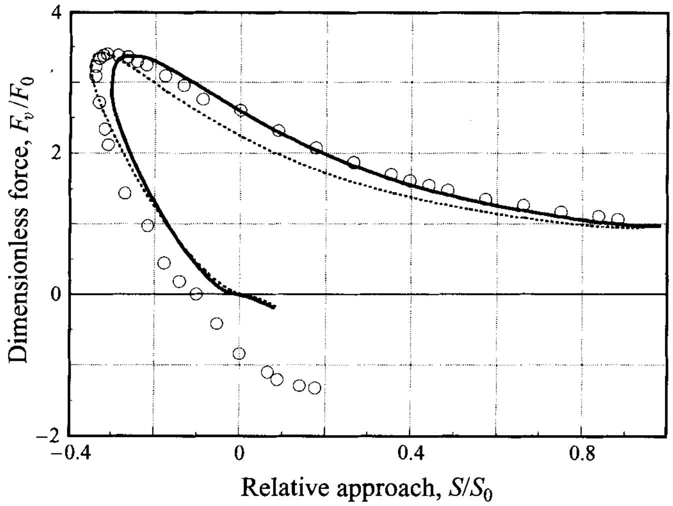

FIGURE 4. Dimensionless force as a function of the relative separation distance between the undeformed surfaces: a comparison of the present model with the DSH (○) solution for the case of $St = 5$ and $\epsilon = 0.01$ and for c_k values of 0.5 (- - -) and 0.25 (——).

with the exact DSH numerical solution for the case of $St = 5$ and $\epsilon = 0.01$. The equivalent Hertzian deformation and undeformed profile are also plotted. The deformation corresponds to $v_0 t/S_0 = 2$ when the flattening reached the maximum and started to relax. It can be seen that for the Hertzian solution, the deformation in the inner region is overestimated, but that in the outer region agrees reasonably well with the DSH solution. When a constant deformation with a c_k value of 0.5 is assumed across the outer region, the deformation in this region is also overestimated. However, when c_k is set to zero, the deformation in the outer region is zero. Decreasing the value of c_k from 0.5 to zero increases the radius of the inner region but decreases the deformation in the outer region. Clearly, adjustment of the value of c_k will not lead to a pressure distribution equal to that computed by Davis *et al.* (1986). However, there should be an intermediate value of c_k that results in similar values of the mean pressure obtained from the integral of the two pressure distributions. For the case of $St = 5$ and $\epsilon = 0.01$, the optimum c_k value is found to be 0.25. With this value the inner region is slightly extended so that the deformation in the major proportion of the outer region is close to the DSH solution but a region of under-estimated deformation in the vicinity of $r = a_1$ is developed. Further calculations show that $c_k = 0.25$ is the optimum value for all combinations of the Stokes number and the elasticity parameter as described below.

In figure 4, the relative hydrodynamic viscous force, F_v/F_0 is plotted as a function of the relative approach, S/S_0, for the case of $\epsilon = 0.01$ and $St = 5$, where F_0 $(= 6\pi\eta v_0(R^*)^2/S_0)$ is the initial viscous force at $t = 0$. It may be seen that, for $c_k = 0.25$, there is a relatively good agreement between the DSH data and those calculated using the current model. However, after the point of maximum deformation, the force is overestimated by the current model. The discrepancy becomes quite significant when the force approaches zero. Unlike the DSH model, that described here allows the stored elastic energy to be fully recovered. This is demonstrated by the observation that both the viscous and elastic forces approach zero at the same point in the trajectory. While for the DSH model it was shown that as the spheres separate, a negative pressure at $r = \infty$ is developed due to the inward flow of fluid in order to fill

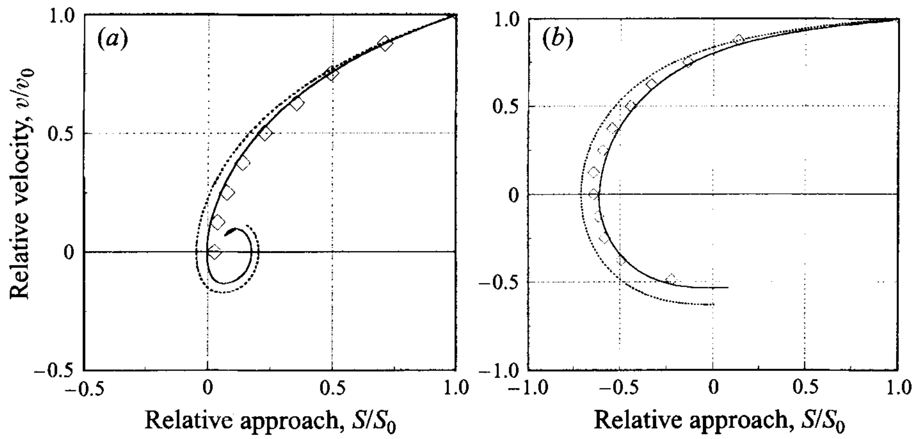

FIGURE 5. Evolution of relative particle velocities predicted by the present model compared with the Wells solution (◇) for $\epsilon = 0.01$ and St equal to (*a*) 2.5 and (*b*) 10 and for c_k values of 0.5 (---) and 0.25 (——).

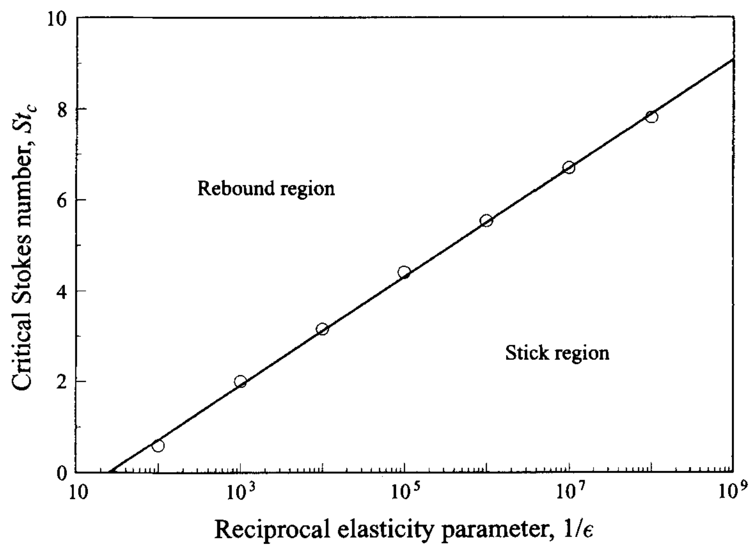

FIGURE 6. The critical Stokes number of rebound as a function of the elasticity parameter predicted by the DSH model (○).

the gap between the receding solid surfaces. As the out-of-balance hydrodynamic fluid force approaches zero, the negative pressure spreads inwards to balance the positive pressure in the inner region. The present model also predicts a negative pressure in the outer region if $v < c_k \dot{\alpha}$. However, for the DSH model, there is a finite elastic deformation when the total force is zero, indicating that the stored elastic energy is not fully released. A residual component of the stored elastic energy was also predicted by Davis *et al.* (1986) when the spheres undergo a damped oscillation and come to rest.

Figure 5 shows the predicted evolution of the relative velocity compared with the Wells (1993) solution for other cases where $\epsilon = 0.01$ with St values of 2.5 and 10 respectively. The figure also indicates, for all cases, that by reducing the scaling coefficient c_k, the viscous damping increases. Davis *et al.* (1986) only provided values

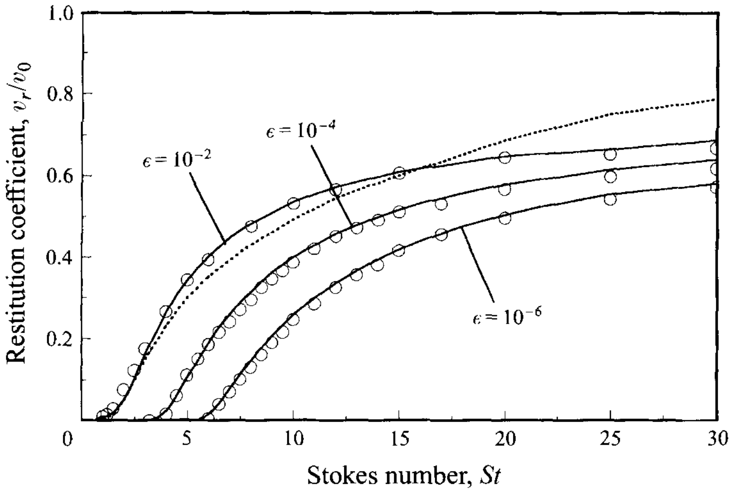

FIGURE 7. Restitution coefficients predicted by the present model with $c_k = 0.25$ (——) compared with those of DSH (○) and Wells (----).

for the maximum rebound velocity. It is also found that the best fit to the maximum rebound velocity of the DSH solution is obtained by choosing $c_k = 0.25$.

Davis *et al.* (1986) reported some results for the restitution coefficients, defined as the ratio of the maximum rebound velocity to the initial impact velocity, for various elasticity parameters and Stokes numbers. It was shown that for a given elasticity parameter there is a critical Stokes number above which rebound occurs. The critical Stokes numbers predicted by the DSH solution for different elasticity parameters are plotted in figure 6. The relationship between the critical Stokes number and elasticity parameter may be described by the following semi-logarithmic function:

$$St_c = 0.518 \ln \frac{1}{25\epsilon}, \tag{30}$$

which provides a mapping of the (St, ϵ)-plane into stick and rebound regions. Barnocky & Davis (1988*b*) developed a similar curve-fitting equation, but the associated parameters are a less precise description of the original data.

In figure 7 the restitution coefficients predicted by the present model using a c_k value of 0.25 are plotted in comparison with those reported by Davis *et al.* (1986) and Wells (1993). It may be seen that the restitution coefficients predicted by the present model agree well with the DSH model. However, the data calculated by Wells underestimates the rebound velocities for Stokes numbers less than 15, while for $St > 15$, the data are overestimated.

Finally, in figures 8(*a*) and 8(*b*), the evolution of the relative particle velocities and forces predicted by the present solution are compared with those calculated for rigid spheres. The present solution was evaluated for the cases when $St = 5$ and $\epsilon = 0.001$ and 0.0001. It may be seen that for $\epsilon = 0.001$, the rigid-sphere solution agrees well with the elastohydrodynamic solution at large separation distances but starts to diverge for small values. The force predicted by the rigid-sphere solution is generally much greater, resulting in an over-damped relative velocity. However, when the elasticity parameter is reduced to $\epsilon = 0.0001$, the difference between the two solutions is very small. Under these circumstances, the simple rigid-sphere solution obtained by Adams & Perchard (1985) applies.

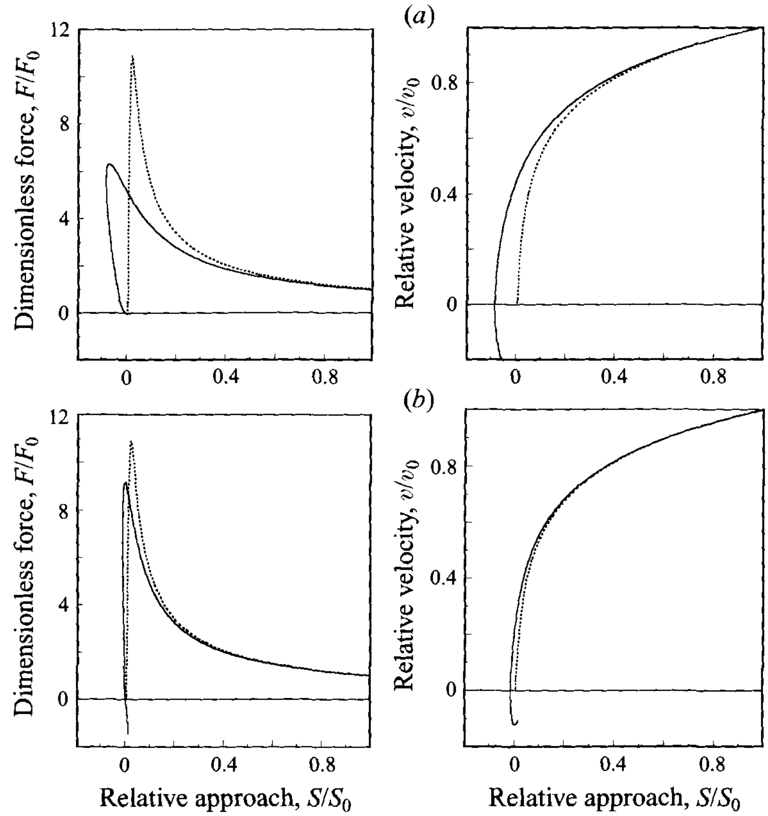

FIGURE 8. Evolution of relative particle velocities and forces for $St = 5$ and (*a*) $\epsilon = 0.001$ and (*b*) $\epsilon = 0.0001$: a comparison of the rigid sphere model (----) and the present elastohydrodynamic model (——).

5. Conclusions

An approximation to the elastohydrodynamic collision between two spherical solids with an interstitial incompressible Newtonian fluid of constant viscosity has been examined. It was shown that the simplified model proposed by Wells (1993) assumes a dynamic deformation that is similar to the truncated Taylor series for the Hertzian profile. This simplification leads to a set of fourth-order ordinary differential equations that can only be solved numerically. If the truncated Taylor series for the exact Hertzian deformation is assumed for the spheres, analytical expressions for the dynamic pressure of the fluid and the viscous force are still not attainable.

An approximate model has been developed by assuming that the elastic deformation in the inner region is described by the Hertzian profile and that outside the inner region is uniform. This is based on the results of the exact numerical solution of Davis *et al.* (1986) who found that the deformation leads to a relatively planar inner region centred on the line of approach, which is very similar in form to a Hertzian deformation. The proposed Hertzian-like deformation leads to a simple closed-form analytical solution for the hydrodynamic viscous force. By using a scaling coefficient value of 0.25, which is equivalent to assuming that the elastic deformation outside the Herzian contact area is one quarter of that at centre, it has been demonstrated that the present analytical solution agrees well with the accurate numerical solution of Davis *et al.* (1986).

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